COLORADO STATE UNIVERSITY

Assignment 2 **Agricultural & Resource Economics / Economics 535** *S.R. Koontz*
Fall 2023 **Applied Econometrics**

This assignment is worth 25 points. Please show your work concisely. A printed spreadsheet page is sufficient for calculations. A hand-written document is sufficient for the other questions. Correct answers that are disorganized and do not have clear supporting discussion are worth little. Communication is important. And so is clarity

1. The following regression equation is a production function where for Q is output, L is labor and K is capital:

$$\ln(Q_t) = 1.035 + \underset{(0.095)}{0.675}\ln(K_t) + \underset{(0.155)}{0.305}\ln(L_t)$$

$R^2 = 0.395$, $\mathrm{Cov}(\hat{\beta}_K, \hat{\beta}_L) = +0.0004$, $T = 33$, and the standard errors are in parenthesis.

Answer the following questions. When appropriate, be sure to state the null hypothesis, calculate the test statistic, state the critical value, and draw the test conclusion. P-values are also very good to report.

a. Give a brief economic interpretation of each slope coefficient.
b. Set up the appropriate hypotheses and perform the tests to determine if each parameter estimate is significantly different from zero.
c. Set up the appropriate hypothesis and perform the test to determine if the parameter estimates associated with labor and capital are equal.
d. Test the null hypothesis that there are constant returns to scale.
e. You conducted small sample tests for parts b-d. Conduct large sample tests where the sample size is large enough to not matter. (Hint: same statistic but different p-value.)
f. Test the null hypothesis that the regression is insignificant.

2. Consider the following regression model: $Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \varepsilon_t$

with sample data:

$n = 20$ $\Sigma X_1 = 350.1$ $\Sigma X_2 = 210$ $\Sigma Y = 559.94$
$\Sigma X_1^2 = 6446.89$ $\Sigma X_2^2 = 2870$ $\Sigma Y^2 = 16057.91$
$\Sigma X_1X_2 = 3457.7$ $\Sigma X_1Y = 9492.559$ $\Sigma X_2Y = 6167.34$.

Refer to Gujarati & Porter Appendix C and especially C.10. Please be sure to set up the matrices and show your work using these. You will need to invert a 3×3 matrix. You can do this by hand, refer to Appendix B, or use software. (Hint: MINVERSE, highlight the cells, F2, ctrl-shift-enter.) Also, correcting for deviations around the mean are only needed for TSS and ESS.

When appropriate, be sure to state the null hypothesis, calculate the test statistic, state the critical value, and draw the test conclusion. Also provide a p-value.

a. Compute the OLS estimates of β_0, β_1, and β_2.
b. Compute the variance-covariance matrix of regression coefficients.
c. Compute R^2 and $\bar{R}^2$.
d. Test the null hypothesis H_0: $\beta_1 = -1$.
e. Test the null hypothesis H_0: $\beta_1 = \beta_2 = 0$.
f. Test the null hypothesis H_0: $\beta_1 = -4\beta_2$.

3. Using the sample from the question above, consider two alternative models:

$$Y_t = \beta_{0S1} + \beta_{1S} X_{1t} + \varepsilon_t \quad \text{and} \quad Y_t = \beta_{0S2} + \beta_{2S} X_{2t} + \varepsilon_t$$

The original model is referred to as the long regression and the two alternative models are short regressions for the obvious reason. Please invert the 2×2 matrices by hand using methods in Appendix B.

a. Compute the OLS estimates of β_{0S1}, β_{1S}, β_{0S2}, and β_{2S}.
b. Are the estimates of the two slope coefficients different between the long and the two short models? Why?

4. A model was estimated using OLS regression with quarterly data from 1990 through 2022 inclusive. There are four independent variables (k=4). The residual sum of squares was 38.5, and the explained sum of squares was 96.0. Answer the following questions.

a. When three seasonal dummy variables were added to the equation and the equation was re-estimated then the explained sum of squares increased to 99.5. Test for the presence of seasonality.
b. Two further regressions, using the original specification, were estimated for subsamples of 1990 through 2007 and 2008 through 2022. The resulting residual sums of squares were 19.5 and 16.5. First, test if the error variances are equal in the two subsamples. Second, test if the coefficients are identical in the two subsamples.

5. Optional: Determine the Eigenvalues and Eigenvectors of $(X'X)$ and $(X'X)^{-1}$ from the sample used in question 2.

1. **a**. Elasticities of production – G&P Section 7.9. You must say, "elasticity."

b. H_0: $\hat{\beta}_j = 0$ where j = K, L. $t = \frac{\hat{\beta}_j}{\sqrt{V(\hat{\beta}_j)}} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$.

Critical value of $t_{30}(\alpha = 5\%/2) = 2.0423$

$t_K = \frac{0.675}{0.095} = 7.1053$. Small sample p-value of <.01%. Large sample p-value of <.01%. Reject.

$t_L = \frac{0.305}{0.155} = 2.0423$. Small sample p-value of 5.84%. Large sample p-value of 4.91%. Reject.

c. H_0: $\hat{\beta}_K = \hat{\beta}_L$. H_0: $\hat{\beta}_K - \hat{\beta}_L = 0$.

$$t = \frac{\hat{\beta}_K - \hat{\beta}_L}{\sqrt{V(\hat{\beta}_K) + V(\hat{\beta}_L) - 2Cov(\hat{\beta}_K,\hat{\beta}_L)}} \text{ so that } t = \frac{0.675 - 0.305}{\sqrt{(0.095)^2 + (0.155)^2 - 2Cov(0.0004)}} = +2.0603$$

Same critical values. Small sample p-value of 4.81%. Reject. Large sample p-value of 3.94%. Reject.

d. H_0: $\hat{\beta}_K + \hat{\beta}_L = 1$.

$$t = \frac{\hat{\beta}_K + \hat{\beta}_L - 1}{\sqrt{V(\hat{\beta}_K) + V(\hat{\beta}_L) + 2Cov(\hat{\beta}_K,\hat{\beta}_L)}} \text{ so that } t = \frac{0.675 + 0.315 - 1}{\sqrt{(0.095)^2 + (0.155)^2 + 2Cov(0.0004)}} = -0.1087$$

Same critical value. Small sample p-value of 91.42%. Fail to reject. Large sample p-value of 91.34%. Fail to reject.

e. Use the normal distribution. Critical value is 1.96 and p-values are smaller.

f. H_0: $\hat{\beta}_K = \hat{\beta}_L = 0$.

$$F = \frac{R^2/k}{(1- R^2)/(N-(k+1))} = \frac{0.395/2}{(1-0.395)/(33-(2+1))} = 9.7934.$$

Critical value of $F_{2,30}(\alpha = 5\%) = 3.2849$ & p-value = 0.0005. Reject, at least one is not.

2. **a**.

$$X'X = \begin{bmatrix} N & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix} = \begin{bmatrix} 20 & 350.1 & 210 \\ 350.1 & 6446.89 & 3457.7 \\ 210 & 3457.7 & 2870 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 1.995365 & -0.084933 & -0.043677 \\ -0.084933 & 0.004054 & -0.001331 \\ -0.043677 & -0.001331 & 0.001941 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} \Sigma \mathrm{Y} \\ \Sigma X_1 Y \\ \Sigma X_2 Y \end{bmatrix} = \begin{bmatrix} 559.94 \\ 9492.559 \\ 6167.34 \end{bmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = \begin{bmatrix} 1.995365 & -0.084933 & -0.043677 \\ -0.084933 & 0.004054 & -0.001331 \\ -0.043677 & -0.001331 & 0.001941 \end{bmatrix} \begin{bmatrix} 559.94 \\ 9492.559 \\ 6167.34 \end{bmatrix} = \begin{bmatrix} 41.6799 \\ -0.8700 \\ 0.1474 \end{bmatrix}$$

b.

$$RSS = Y'Y - \hat{\beta}'X'Y = 16057.91 - \begin{bmatrix} 41.6799 & -0.8700 & 0.1474 \end{bmatrix} \begin{bmatrix} 559.94 \\ 9492.559 \\ 6167.34 \end{bmatrix} = 69.8183$$

$$\hat{\sigma}^2 = RSS \,/\, (N + (k+1)) = 69.8183 \,/\, (20 - (2 + 14)) = 4.1070$$

$$V(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$$

$$= 4.1070 \begin{bmatrix} 1.995365 & -0.084933 & -0.043677 \\ -0.084933 & 0.004054 & -0.001331 \\ -0.043677 & -0.001331 & 0.001941 \end{bmatrix} = \begin{bmatrix} 8.1949 & -0.3488 & -0.1794 \\ -0.3488 & 0.0166 & 0.0055 \\ -0.1794 & 0.0055 & 0.0080 \end{bmatrix}$$

c.

$$TSS = Y'Y - N\bar{Y}^2 = 16057.91 - 20\left(\frac{559.94}{20}\right)^2 = 381.265$$

R^2 = ESS/TSS = (TSS – RSS)/TSS = (381.265 – 69.8183)/381.265 = 0.8169

$$\bar{R}^2 = 1 - (1 - R^2)\frac{N-1}{N-(k+1)} = 1 - (1 - 0.8169)\frac{20-1}{20-(2+1)} = 0.7953$$

d. H_0: $\hat{\beta}_1 = -1$.

$t = \frac{\hat{\beta}_1-(-1)}{\sqrt{V(\hat{\beta}_1)}}$ so that $t = \frac{-0.8700+1}{0.1290} = 1.0072$.

$t_{17}(\alpha = 5\%/2) = 2.1098$ & p-value = 0.3280. Fail to reject.

e. H_0: $\hat{\beta}_1 = \hat{\beta}_2 = 0$.

$$F = \frac{R^2 / k}{(1 - R^2) / (N - (k+1))} = \frac{0.8169 / 2}{(1 - 0.8169)/ (20 - (2+1))} = 37.9169$$
.

$F_{2,17}(\alpha = 5\%) = 3.5915$ & p-value = <.0001. Reject, at least one not.

f. H_0: $\hat{\beta}_1 = -3\hat{\beta}_2$. H_0: $\hat{\beta}_1 + 3\hat{\beta}_2 = 0$.

$$t = \frac{\hat{\beta}_1 + 3\hat{\beta}_2}{\sqrt{V(\hat{\beta}_1) + 3^2 V(\hat{\beta}_2) + 2 \times 3Cov(\hat{\beta}_1, \hat{\beta}_2)}}$$

so that

$$t = \frac{-0.8700 + 3(0.1474)}{\sqrt{(0.1290)^2 + 9(0.0893)^2 + 6Cov(0.0055)}} = -1.2294$$
.

$t_{17}(\alpha = 5\%/2) = 2.1098$ & p-value = 0.2378. Fail to reject.

3. **a**.

$$X'X = \begin{bmatrix} N & \Sigma X_1 \\ \Sigma X_1 & \Sigma X_1^2 \end{bmatrix} = \begin{bmatrix} 20 & 350.1 \\ 350.1 & 6446.89 \end{bmatrix}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}}{a_{11}a_{22}-(a_{12}a_{21})}$.

$$(X'X)^{-1} = \begin{bmatrix} 1.012422 & -0.05498 \\ -0.05498 & 0.003141 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} \Sigma Y \\ \Sigma X_1 Y \end{bmatrix} = \begin{bmatrix} 559.94 \\ 9492.559 \end{bmatrix}$$

$$\hat{\beta}_{S1} = (X'X)^{-1}X'Y = \begin{bmatrix} 1.012422 & -0.05498 \\ -0.05498 & 0.003141 \end{bmatrix} \begin{bmatrix} 469.223 \\ 5422.702 \end{bmatrix} = \begin{bmatrix} 44.9963 \\ -0.9711 \end{bmatrix}$$

$$X'X = \begin{bmatrix} N & \Sigma X_2 \\ \Sigma X_2 & \Sigma X_2^2 \end{bmatrix} = \begin{bmatrix} 20 & 210 \\ 210 & 2870 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 0.215789 & -0.01579 \\ -0.015789 & 0.001504 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} \Sigma Y \\ \Sigma X_2 Y \end{bmatrix} = \begin{bmatrix} 559.94 \\ 6167.34 \end{bmatrix}$$

$$\hat{\beta}_{S2} = (X'X)^{-1}X'Y = \begin{bmatrix} 0.215789 & -0.01579 \\ -0.015789 & 0.001504 \end{bmatrix} \begin{bmatrix} 559.94 \\ 6167.34 \end{bmatrix} = \begin{bmatrix} 23.4501 \\ 0.4330 \end{bmatrix}$$

b.

Coeff	Long	Short1	Short2
β_0	41.6799	44.9963	23.4501
β_1	-0.8700	-0.9711	...
β_2	0.1474	…	0.4330

Not the same only because of $\Sigma X_1 X_2 \neq 0$. The short regressions do not use this piece but the long regression does. If this piece was zero then the long and the short slopes would be the same.

Intercepts are always different if the combination of means are different.

4. Chow Test – G&P Section 8.7. ESS = 96.0 and RSS = 38.5 so TSS = 134.5. N = 132.

a. With 3 more variables EES = 99.5 or RSS = 35.0.

H_0: $\beta_6 = \beta_7 = \beta_8 = 0$

$$F = \frac{(RSS_{new} - RSS_{old})/m}{RSS_{new}/df} = \frac{(RSS_{new} - RSS_{old})/m}{RSS_{new}/df} = \frac{(38.5 - 35.0)/3}{35.0/(132 - (7 + 1))} = 4.1333$$

Under null hypothesis F is distributed $F_{3,124}$ and the 5% critical value is 2.6777. P-value = 0.0079. Reject null. Seasonality is significant.

b. First, H_0: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$.

$$F = \frac{RSS_2/(N_2 - (k + 1))}{RSS_1/(N_1 - (k + 1))} = \frac{16.5/(60 - (4 + 1))}{19.5/(72 - (4 + 1))} = \frac{0.3000}{0.29104} = 1.0308$$

Under null hypothesis F is distributed $F_{55,67}$ and the 5% critical value is 1.5242. P-value = 0.4499. Fail to reject. The variances in the two periods are the same.

Second, H_0: $\hat{\beta}_{01} = \hat{\beta}_{02}, \hat{\beta}_{11} = \hat{\beta}_{12}, \hat{\beta}_{21} = \hat{\beta}_{22}, \hat{\beta}_{31} = \hat{\beta}_{32}, \hat{\beta}_{41} = \hat{\beta}_{42}$. So that m=5.

$$F = \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(N_1 + N_2 - 2(k + 1))} = \frac{(28.5 - (19.5 + 16.5)/5}{(19.5 + 16.5)/(72 + 60 - (2 \times 5))} = 1.6944$$

Under null hypothesis F is distributed $F_{5,122}$ and the 5% critical value is 2.2886. P-value = 0.1409. Fail to reject. No significant structural change.

5.

$X'X$: $\Lambda = \begin{bmatrix} 8570.7788 \\ 765.61123 \\ 0.5000158 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0.0476892 & 0.0034247 & 0.9988564 \\ 0.8532667 & 0.5192896 & -0.04252 \\ 0.5192896 & -0.854319 & -0.021864 \end{bmatrix}$

$X'X^{-1}$: $\Lambda = \begin{bmatrix} 1.9999368 \\ 0.0013061 \\ 0.0001167 \end{bmatrix}$ $\quad C = \begin{bmatrix} -0.998856 & 0.0034247 & 0.0476892 \\ 0.0013061 & 0.5197384 & 0.8532667 \\ 0.0001167 & -0.854319 & 0.5192896 \end{bmatrix}$